5 Deriving the wave equation

From now on I consider only linear *second order* partial differential equations, and the first equation I will study is the so-called *wave equation* which, in one spatial dimension, has the form

$$u_{tt} = c^2 u_{xx}, \tag{5.1}$$

where c is some constant.

I will present one possible way to arrive to this equation noting that exactly the same equation appears in many other physical situations. Recall that I obtained the transport equation in the previous lectures as the consequence of the fundamental conservation law. The wave equation is the consequence of another fundamental physical law: *the second Newton's law*, that states that the product of mass and the acceleration is equal to the net force applied to the body.

Consider, for example, the classical mechanical system of mass on a spring (see Fig. 1). If the only

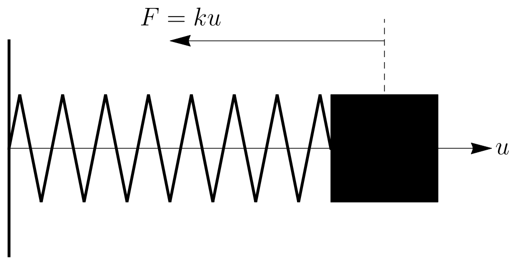

Figure 1: Mass on a spring.

force acting on the mass is the restoring force of the spring then, using another physical law, namely *Hook's law*, I arrive at

$$mu'' = -ku.$$

Here $u(t)$ is the position of my mass at time t, $u(0) = 0$ is the equilibrium position, $u''(t)$ is the acceleration, m is the mass, ku is the Hook law that says that the force is proportional to the displacement (note that this is actually true only for small displacements), the minus because the force points opposite to the u-axis, and k is the parameter, rigidity of the spring. Rewriting this as

$$u'' + \omega^2 u = 0, \quad \omega = \sqrt{\frac{k}{m}},$$

I find that the general solution to this ODE can be written as

$$u(t) = C_1 \cos \omega t + C_2 \sin \omega t,$$

which represents *harmonic oscillations.* C_1 and C_2 are arbitrary constants that can be uniquely identified using the initial position and initial velocity of the mass.

Now let me consider a more general situation, when I have $n+1$ equal masses, such that the zeroth and the last one are fixed, but all others are free to move along the axis and linearly interconnected with the springs with the same rigidity k (see Fig. 2).

Math 483/683: Partial Differential Equations by Artem Novozhilov
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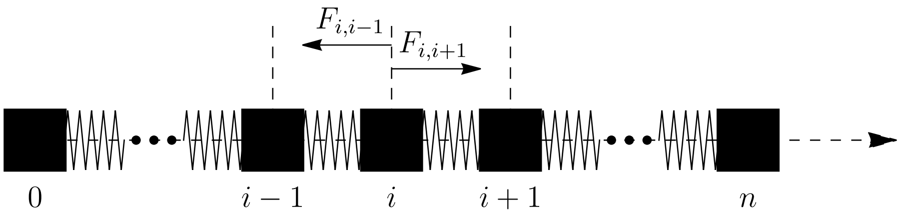

Figure 2: A system of $n+1$ masses connected by identical springs. The initial and final masses are fixed.

Let $u_i(t)$ denote the displacement of the i-th mass from the equilibrium position x_i. Using again the second Newton's law I get, for the i-th mass

$$mu_i'' = F_{i,i+1} - F_{i,i-1} = k(u_{i+1}-u_i) - k(u_i - u_{i-1}) = k(u_{i+1} - 2u_i + u_{i-1}),\ i = 1, \ldots, n-1,\ u_0 = 0,\ u_n = 0.$$

For each u_i I also should have two initial conditions: initial displacement and initial velocity. This is still a system of $(n-1)$ ODE, which can be actually analyzed and solved. I, however, instead of solving it, will consider the situation when the number of my masses grow to infinity, i.e., $n \to \infty$. Assuming that at the equilibrium all my masses separated by the same distance h it is equivalently to say that I consider the case when $h \to 0$. In other words I consider a continuous limit of a discrete system!

I have that $u_i(t) = u(t, x_i)$ and the equilibrium positions of all the masses become a continuous variable x and my discrete system of masses turns into a continuous rod. That is, when $h \to 0$, I have that the vector $(u_1(t), \ldots, u_i(t), \ldots, u_{n-1}(t))$ becomes a function of two variables $u(t, x)$, where now the meaning of $u(t, x)$ is the displacement of the section of a rod at time t that had the coordinate x at rest. To carefully perform the limit I also need to understand how my two parameters m and k behave. It is natural to assume that

$$m = \rho S h,$$

where ρ is density of the rod and S is the area of the transverse section. For the rigidity I will use the physical fact that

$$k = \frac{ES}{h}\,,$$

where E is the *Young modulus*. Hence I get that my ODE system can be rewritten now as

$$u_i'' = \frac{E}{\rho}\frac{(u_{i+1} - 2u_i + u_{i-1})}{h^2}\,.$$

The left hand side tends to u_{tt} as $h \to 0$. To understand what happens on the right hand side consider Taylor's series for u_{i+1}, u_i, u_{i-1} around $h = 0$:

$$\begin{aligned} u_{i+1}(t) &= u(x_i + h, t) = u(x_i, t) + u_x'(x_i, t)h + \frac{1}{2!}u_{xx}''(x_i, t)h^2 + \ldots, \\ u_i(t) &= u(x_i, t), \\ u_{i-1}(t) &= u(x_i - h, t) = u(x_i, t) - u_x'(x_i, t)h + \frac{1}{2!}u_{xx}''(x_i, t)h^2 + \ldots, \end{aligned}$$

where the dots denote the terms of order 3 and above in variable h. Now if I plug these expressions in my differential equation, cancel h^2, I can see that

$$\frac{(u_{i+1}-2u_i+u_{i-1})}{h^2}(t)=u_{xx}(x_i,t)+\ldots,$$

where now the dots denote the terms that of order 1 in variable h and hence approach 0 as $h\to 0$. Hence

$$\frac{(u_{i+1}-2u_i+u_{i-1})}{h^2}(t)\to u_{xx}(x,t),\quad h\to 0.$$

Finally, I can conclude that the continuous limit of my discrete system of masses on the springs is described by the *wave equation*

$$u_{tt}=c^2u_{xx},\quad c=\sqrt{\frac{E}{\rho}}\,.$$

I also have, by the same limit procedure, that I need two initial conditions

$$u(0,x)=f(x),\quad u_t(0,x)=g(x),$$

and, if the ends of my rod are fixed, two boundary conditions

$$u(t,0)=u(t,l)=0,$$

where l is the length of the rod.

Moreover, if I consider a more complicated system of masses (say, on a plane, where each mass has four neighbors, on in the three dimensional space, where each mass has 6 neighbors), very similar reasonings lead to two- and three- dimensional wave equations

$$u_{tt}=c^2\Delta u,$$

where Δ is the Laplace operator, i.e., $\Delta u=u_{xx}+u_{yy}$ on the plane, and $\Delta u=u_{xx}+u_{yy}+u_{zz}$ in the three dimensional space.

Exactly the same wave equation appears in many situations where some wave processes occur, such as sound waves, light waves, electric waves, water waves, etc. In this lecture I showed that the longitudinal oscillations of a continuous rod are described by the wave equation. It can be shown that (small) transversal oscillations of a string (such as guitar or violin string) are also described by the same equation. To see this heuristically, recall that the second derivative of a function geometrically tells us whether the graph of this function is convex (the second derivative is positive) or concave (the second derivative is negative). Since the physical meaning of the second time derivative is the acceleration, you can see now that the equation $u_{tt}=c^2u_{xx}$ tells us that the acceleration is negative (which pushes the string down) if the form of my string is concave, and positive (which pushes the string up) is the string shape is convex, as should be intuitively expected from a string of some musical instrument.

5.1 Test yourself

5.1. Formulate the second Newton's law.

5.2. Formulate Hook's law.

5.3. How many initial conditions does the wave equation require?

5.4. Let $y = f(x)$ be a smooth function. Recall the form of its Taylor series around the point x_0. What are Taylor's series for $e^x, \sin x, \cos x, \frac{1}{1-x}$ around $x_0 = 0$? Can you recall their radii of convergence?

5.5. Using the notations of this section, what would be the limits

$$\frac{u_{i+1} - u_i}{h}, \quad \frac{u_i - u_{i-1}}{h}, \quad \frac{u_{i+1} - u_{i-1}}{2h}, \quad \text{as } h \to 0?$$